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Some Convergence Results for Padé Approximants

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The primary objective of this paper is to investigate the convergence of sequences of Padé approximants for meromorphic functions of the form

$$F(x) = \sum_{i=1}^{\infty} \frac{A_i}{a_i - x}.$$

In Theorem 6 we give simple conditions on the A_i and a_i which imply convergence of diagonal sequences which lie above the principal diagonal in the Padé table. We conclude that certain P-fractions associated with $F(x)$ converge uniformly on compact sets bounded away from the poles.

In the first part of the paper we show that Padé approximation may be viewed, from an algebraic point of view, as a theory of best rational approximation in a field of formal power series. The approximation is relative to a discrete valuation. This point of view stresses the many similarities with the theory of best rational approximation of real numbers as encountered in number theory [1, p. 154].

BEST APPROXIMATION AND P-FRACTIONS

Let $\mathfrak{F}$ denote the family of formal power series of the form

$$F(x) = c_0 x^{-N} + c_1 x^{-N+1} + \cdots + c_{N-1} x^{-1} + c_N + \cdots, \tag{1}$$

with complex coefficients. $\mathfrak{F}$ is a field and we may view the field $\mathfrak{R}$ of rational functions, as imbedded in $\mathfrak{F}$. In fact $\mathfrak{F}$ is the completion of $\mathfrak{R}$ relative to the metric $\rho(F, G) = \varphi(F - G)$, where φ is the discrete valuation

$$\varphi(F) = \begin{cases} 2^N & \text{if } F(x) = c_0 x^{-N} + c_1 x^{-N+1} + \cdots,\ c_0 \neq 0 \\ 0 & \text{if } F(x) = 0. \end{cases}$$

The following theorem is easily proved in a manner similar to Theorem 5.1 in [3].

THEOREM 1. *For every $F \in \mathfrak{F}$ and for every pair (m, n) of nonnegative integers, there exists a unique rational function of the form*

$$F_{m,n}(x) = P_{m,n}(x)/x^N Q_{m,n}(x),$$

where $P_{m,n}(x)$ and $Q_{m,n}(x)$ are polynomials satisfying:

(i) $\deg P_{m,n} \leqslant n$,

(ii) $\deg Q_{m,n} \leqslant m$, *and*

(iii) *for all polynomials P, Q satisfying* (i) *and* (ii), *respectively, we have*

$$\rho(F, P_{m,n}/x^N Q_{m,n}) \leqslant \rho(F, P/x^N Q).$$

Furthermore, $\rho(F, F_{m,n}) \leqslant \varphi(F) \min\{2^{-m}, 2^{-n}\}$ and hence for every sequence of distinct pairs (m_i, n_i), $F_{m_i,n_i} \to F$ in the sense of the metric ρ.

The polynomials $P_{m,n}$ and $Q_{m,n}$ are normalized so that they are relatively prime and $Q_{m,n}(0) = 1$. The rational function $F_{m,n}$ is called the *(m, n)-Padé approximant* of F. We refer to convergence in the sense of the metric ρ as *formal convergence.*

We will now describe in terms of the valuation φ the algorithm for generating the P-fraction as given by Magnus in [2].

Let $F \in \mathfrak{F}$ and set $b_0 = c_0 x^{-N} + \cdots + c_{N-1} x^{-1} + c_N$. We think of b_0 as the "integral part" of F and $F_0^* = F - b_0$ as the "fractional part." Indeed, $\varphi(b_0) \geqslant 1$ and $\varphi(F_0^*) < 1$. Now if $F_0^* \neq 0$, it has a formal reciprocal in $\mathfrak{F}$ of the form $F_1 = c_0^{(1)} x^{-N_1} + c_1^{(1)} x^{-N_1+1} + \cdots$. Similarly we define b_1 and F_1^*. If $F_1^* \neq 0$ we may write in $\mathfrak{F}$

$$F = b_0 + (b_1 + (1/F_2))^{-1},$$

where F_2 has the form $c_0^{(2)} x^{-N_2} + c_1^{(2)} x^{-N_2+1} + \cdots$. Continuing the process inductively one obtains a finite or infinite sequence of b_n, which are polynomials in $1/x$. For the P-fraction thus derived, Magnus [2] proved the following interesting facts;

(a) The P-fraction for the series $x^k F(x)$, when $k \leqslant N$, generates all the distinct rational functions in the sequence $\{x^k F_{m,N-k+m} : m = 0, 1, 2,...\}$ and, when $k > N$, it generates the distinct members of $\{x^k F_{k-N+m,m} : m = 0, 1, 2,...\}$. It follows that the P-fractions generate all the Padé approximants of F.

(b) The P-fractions for F terminates if and only if F corresponds to a rational function.

REPRESENTATIONS FOR $F_{m,n}(x)$

Let $\Delta_{m,n}$ denote the persymmetric determinant $| c_{n+i-j} |$ where $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant m$ and let $\Delta_{m,n}(x)$ and $\Delta^{i}_{m,n}$ denote the corresponding determinants obtained from $\Delta_{m,n}$ by replacing the first row by $1, x,..., x^m$ and $c_{n+m+i} ,..., c_{n+i}$, respectively. The following formulas are established as in [3, p. 243];

$$\text{If} \quad \Delta_{m-1,n} \neq 0, \quad \text{then} \quad Q_{m,n}(x) = \Delta_{m,n}(x)/\Delta_{m-1,n}, \tag{2}$$

and

$$F(x)\, x^N Q_{m,n}(x) - P_{m,n}(x) = (\Delta_{m-1,n})^{-1} \sum_{i=0}^{\infty} \Delta^{(i)}_{m,n} x^{m+n+i}. \tag{3}$$

It follows from (3) that

$$\begin{aligned} P_{m,n+1}Q_{m,n} - P_{m,n}Q_{m,n+1} &= P_{m+1,n+1}Q_{m,n} - P_{m,n}Q_{m+1,n+1} \\ &= (-1)^m (\Delta_{m,n+1}/\Delta_{m-1,n})\, x^{m+n+1}. \end{aligned} \tag{4}$$

In view of (4), if $\Delta_{m-1,n} \neq 0$, for $n \geqslant n_0$ and m fixed,

$$F_{m,n}(x) = F_{m,n_0}(x) + (-1)^m \sum_{i=n_0}^{n-1} \frac{\Delta_{m,i+1}}{\Delta_{m-1,i}} \frac{x^{-N+m+i+1}}{Q_{m,i}(x)\, Q_{m,i+1}(x)}, \tag{5}$$

and if $\Delta_{i,i+p} \neq 0$ for $i \geqslant m_0$ and $p \geqslant 0$ fixed,

$$F_{m,m+p}(x) = F_{m_0,m_0+p}(x) + \sum_{i=m_0}^{m-1} (-1)^i \frac{\Delta_{i,i+p+1}}{\Delta_{i-1,i+p}} \frac{x^{-N+p+2i+1}}{Q_{i,i+p}(x)\, Q_{i+1,i+p+1}(x)}. \tag{6}$$

The rational functions in (5) and (6) lie, respectively, in the n-th row and the p-th diagonal file of the Padé table of F. The latter correspond to the approximants of certain P-fractions.

MAIN RESULTS

In this section we use the representations (5) and (6) to study the convergences of rows and diagonal files of Padé approximants for functions of the form

$$F(x) = \sum_{i=1}^{\infty} \frac{A_i}{a_i - x}, \tag{7}$$

where it is assumed that

$$0 < | a_1 | < | a_2 | < \cdots, \quad \sum_{i=1}^{\infty} \left| \frac{A_i}{a_i} \right| < \infty, \qquad \text{and} \qquad \sum_{i=1}^{\infty} \frac{1}{| a_i |} < \infty.$$

For convenience, let $S(m)$ denote the set of m-tuples $k = (k_1, k_2, \dots, k_m)$ of nonnegative integers with $k_1 < k_2 < \cdots < k_m$, and, also let $K = (1, 2, 3, \dots, m)$. Furthermore, for each pair (m, n) and $k \in S(m)$, let

$$T^k_{m,n} = A_{k_1} A_{k_2} \cdots A_{k_m} (a_{k_1} a_{k_2} \cdots a_{k_m})^{-(m+n)} \prod_{1 \leqslant i < j \leqslant m} (a_{k_j} - a_{k_i})^2$$

and (8)

$$T^k_{m,n}(x) = T^k_{m,n} \prod_{\nu=1}^{m} \left(1 - \frac{x}{a_{k_\nu}}\right).$$

LEMMA 1. *For $F(x)$ of the form* (7) *and for $n \geqslant m$, we have*

$$\Delta_{m-1,n} = (-1)^{m(m-1)/2} \sum_{k \in S(m)} T^k_{m,n}$$

$$\Delta_{m,n}(x) = (-1)^{m(m-1)/2} \sum_{k \in S(m)} T^k_{m,n}(x). \tag{9}$$

Proof. If $F(x) = \sum_{n=0}^{\infty} c_n x^n$ it is easy to see that $c_n = \sum_{i=1}^{\infty} A_i a_i^{-n-1}$. Substituting these into $\Delta_{m-1,n}$ yields,

$$\Delta_{m-1,n} = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_m=1}^{\infty} A_{k_1} A_{k_2} \cdots A_{k_m}$$

$$\times\, a_{k_1}^{-n-1} a_{k_2}^{-n-2} \cdots a_{k_m}^{-n-m} \begin{vmatrix} a_{k_1} & \cdots & a_{k_1}^{m-1} \\ & \cdots & \\ a_{k_m} & \cdots & a_{k_m}^{m-1} \end{vmatrix}$$

$$= \sum_{\substack{k_i \neq k_j \\ i \neq j}} A_{k_1} A_{k_2} \cdots A_{k_m} (a_{k_1} a_{k_2} \cdots a_{k_m})^{-(n+m)}$$

$$\times\, a_{k_1}^{m-1} a_{k_2}^{m-2} \cdots a_{k_m}^{0} \prod_{1 \leqslant i < j \leqslant m} (a_{k_i} - a_{k_j}).$$

The sum of the $m!$ terms in this series which involves the indices $k_1 < k_2 < \cdots < k_m$ is

$$A_{k_1} \cdots A_{k_m} (a_{k_1} \cdots a_{k_m})^{-(n+m)} \prod_{1 \leqslant i < j \leqslant m} (a_{k_j} - a_{k_i}) \sum \pm a_{k_1'}^{m-1} a_{k_2'}^{m-2} \cdots a_{k_1'}^{0}$$

$$= (-1)^{m(m-1)/2} A_{k_1} \cdots A_{k_m} (a_{k_1} \cdots a_{k_m})^{-(n+m)} \prod_{1 \leqslant i < j \leqslant m} (a_{k_j} - a_{k_i})^2$$

$$= (-1)^{m(m-1)/2}\, T^k_{m,n}\,.$$

This proves the first assertion of the lemma and the second is proved in a similar manner.

We shall say that $F(x)$ satisfies condition D_p provided

$$\lim_{m\to\infty} \frac{\sum_{k\in S(m)} | T^k_{m,m+p} |}{| T^K_{m,m+p} |} = 1,$$

i.e., the leading term of the sum $\sum T^k_{m,n}$ is dominant on the p-th diagonal file. We can now state our main theorem.

THEOREM 2. *Let $F(x)$ satisfy condition D_p for $p = 0$. Then for each $p \geqslant 0$ we have,*

(i) $| \Delta_{m,m+p+1}/\Delta_{m-1,m+p} | \leqslant | a_1 a_2 \cdots a_m/2 \cdot 2 \cdots 2 |^{-2}$ *for all sufficiently large m,*

(ii) $\lim_{m\to\infty} Q_{m,m+p}(x) = \prod_{\nu=1}^{\infty} (1 - x/a_\nu)$ *uniformly on any bounded set, and*

(iii) $\lim_{m\to\infty} F_{m,m+p}(x) = F(x)$, *the convergence being uniform on any compact set bounded away from the poles of F.*

COROLLARY 1. *Let $F(x)$ be as in the theorem. Then the P-fractions corresponding to the series $x^{-k}F(x)$, for each $k \geqslant 0$, converge uniformly to $x^{-k}F(x)$ on any compact set bounded away from the poles.*

Before proving Theorem 2 we state a corresponding theorem for rows of the Padé table. This theorem gives the primary motivation for the condition D_p and most of its content can be found in Perron [3, pp. 265–270]. We shall, however, sketch a simplified proof based on the formulas developed above for functions of the form (7).

THEOREM 3. *Let $F(x)$ be of the form* (7). *Then for each $m \geqslant 0$ we have*

(i) $\lim_{n\to\infty} \sum_{k\in S(m)} | T^k_{m,n} |/| T^K_{m,n} | = 1$,

(ii) $\lim_{n\to\infty} | \Delta_{m,n+1}/\Delta_{m-1,n} |^{1/n} = | a_{m+1} |^{-1}$,

(iii) $\lim_{n\to\infty} Q_{m,n}(x) = \prod_{\nu=1}^{m} (1 - x/a_\nu)$ *uniformly,*

(iv) $\lim_{n\to\infty} F_{m,n}(x) = F(x)$, *the convergence being uniform in the region defined by $| x | \leqslant | a_{m+1} | - \epsilon$, $| x - a_i | \geqslant \epsilon > 0$, $i = 1, 2,..., m$. Furthermore, the sequence diverges for $| x | > | a_{m+1} |$.*

Proof of Theorem 2. From the definition of $T^k_{m,m+p}$ and condition D_p we have

$$\frac{\sum_{k\neq K} | T^k_{m,m+p} |}{| T^K_{m,m+p} |} = \frac{\sum_{k\neq K} | T^k_{m,m} | \left| \dfrac{a_1 a_2 \cdots a_m}{a_{k_1} a_{k_2} \cdots a_{k_m}} \right|^p}{| T^K_{m,m} |} \leqslant \frac{\sum_{k\neq K} | T^k_{m,m} |}{| T^K_{m,m} |} \to 0$$

as $m \to \infty$. Hence for sufficiently large m,

$$\sum_{k \neq K} | T^k_{m,m+p} | \leqslant \tfrac{1}{2} | T^K_{m,m+p} | \tag{10}$$

and hence by (9), $| \Delta_{m-1,m+p} | \geqslant$

$$| T^K_{m,m+p} | - \sum_{k \neq K} | T^k_{m,m+p} | \geqslant \tfrac{1}{2} | T^K_{m,m+p} | > 0. \tag{11}$$

Therefore,

$$\sum_{k \neq K} | T^K_{m,m+p} | \Big/ | \Delta_{m-1,m+p} | \leqslant 2 \sum_{k \neq K} | T^k_{m,m+p} | \Big/ | T^K_{m,m+p} | \xrightarrow[m\to\infty]{} 0.$$

It follows from this and Lemma 1, that

$$\begin{aligned}
&\Big| Q_{m,m+p}(x) - \prod_{\nu=1}^{m} \left(1 - \frac{x}{a_\nu}\right)\Big| \\
&\quad = \Big| \frac{(-1)^{m(m-1)/2}}{\Delta_{m-1,m+p}} \sum_{k \neq K} T^k_{m,m+p} \Big[\prod_{\nu=1}^{m} \left(1 - \frac{x}{a_{k_\nu}}\right) - \prod_{\nu=1}^{m} \left(1 - \frac{x}{a_\nu}\right)\Big]\Big| \\
&\quad \leqslant 2 \frac{\sum_{k \neq K} | T^k_{m,m+p} |}{| \Delta_{m-1,m+p} |} \prod_{\nu=1}^{\infty} \left(1 + \frac{R}{| a_\nu |}\right) \xrightarrow[m\to\infty]{} 0,
\end{aligned}$$

for all $| x | \leqslant R$. This proves (ii).

From (9) and (10), with p replaced by $p + 1$, we see that $| \Delta_{m,m+p+1} | \leqslant \tfrac{3}{2} | T^K_{m+1,m+p+1} |$ and hence on combining with (11) yields

$$\begin{aligned}
&| \Delta_{m,m+p+1}/\Delta_{m-1,m+p} | \\
&\quad \leqslant \tfrac{3}{2} | T^K_{m,m+p+1} | / \tfrac{1}{2} | T^K_{m,m+p} | \\
&\quad = 3 \, | A_{m+1}/a_{m+1} | \, | a_{m+1} |^{-p-1} \, | a_1 a_2 \cdots a_m |^{-2} \prod_{i=1}^{m} | 1 - (a_i/a_{m+1}) |^2 \\
&\quad \leqslant | a_1 a_2 \cdots a_m / 2 \cdot 2 \cdots 2 |^{-2},
\end{aligned}$$

for sufficiently large m. This proves (i).

Now let D denote a bounded connected region containing the origin and which is bounded away from the poles of $F(x)$. By the uniform convergence in (ii), there exists $\delta > 0$ and an index m_0 such that if $m \geqslant m_0$, $| Q_{m,m+p}(x)| \geqslant \delta$ for all $x \in D$. Let $R = \operatorname{diam}(D)$ and choose $m_1 \geqslant m_0$ such that $| a_i | > 4R$ for all $i \geqslant m_1$. Then for all $x \in D$,

$$\Big| \frac{\Delta_{i,i+p+1}}{\Delta_{i-1,i+p}} \frac{x^{p+2i+1}}{Q_{i,i+p} Q_{i+1,i+p+1}} \Big| \leqslant \frac{R^{p+1}}{\delta^2} \Big| \frac{2R}{a_1} \frac{2R}{a_2} \cdots \frac{2R}{a_i} \Big|^2 \leqslant \frac{L}{4^i}.$$

It follows from this and the representation (6) that the sequence $\{F_{m,m+p}(x): m = 0, 1, 2, \ldots\}$ converges uniformly in D to some holomorphic function $G(x)$. To see that $G(x) \equiv F(x)$ in D observe that from (3) it follows that for sufficiently large m,

$$F_{m,m+p}(x) = c_0 + c_1 x + \cdots + c_{2m+p} x^{2m+p} + d_{2m+p+1} x^{2m+p+1} + \cdots$$

is some neighborhood of the origin. Hence $F^{(n)}_{m,m+p}(0) = n! c_n$ for $m > n/2$. By the uniform convergence we conclude that

$$F^{(n)}_{m,m+p}(0) \xrightarrow[m\to\infty]{} G^{(n)}(0) \qquad \text{so} \qquad G^{(n)}(0) = n!\, c_n \qquad \text{for every } n.$$

Hence, $F(x) \equiv G(x)$ in D. This proves the theorem.

Proof of Theorem 3. Using the identity

$$(a_{k_1} a_{k_2} \cdots a_{k_m})^{-2m+2} \prod_{1 \leqslant i < j \leqslant m} (a_{k_j} - a_{k_i})^2 = \prod_{1 \leqslant i < j \leqslant m} (a_{k_j}^{-1} - a_{k_i}^{-1})^2, \tag{12}$$

it is easy to see that

$$\frac{\sum_{k \neq K} | T^k_{m,n} |}{| T^K_{m,n} |} = \frac{\left(\sum_{k \neq K} \left| \frac{A_{k_1} \cdots A_{k_m}}{a^2_{k_1} \cdots a^2_{k_m}} \right| \prod_{1 \leqslant i < j \leqslant m} | a_{k_j}^{-1} - a_{k_i}^{-1} |^2 \times \left| \frac{a_1 \cdots a_m}{a_{k_1} \cdots a_{k_m}} \right|^{n-m} \right)}{\left| \frac{A_1 \cdots A_m}{{a_1}^2 \cdots {a_m}^2} \right| \prod_{1 \leqslant i < j \leqslant m} | a_j^{-1} - a_i^{-1} |^2}$$

$$\leqslant M \left| \frac{a_m}{a_{m+1}} \right|^{n-m} \to 0 \quad \text{as} \quad n \to \infty. \tag{13}$$

This proves (i).

Using this, we get

$$\big| |(\Delta_{m-1,n}/T^K_{m,n})| - 1 \big| \leqslant |(\Delta_{m-1,n}/T^K_{m,n}) - 1 | \leqslant \sum_{k \neq K} | T^k_{m,n} |/| T^K_{m,n} | \xrightarrow[n\to\infty]{} 0, \tag{14}$$

and since

$$| T^K_{m+1,n+1}/T^K_{m,n} | = c \, | a_{m+1} |^{-n},$$

where c is a nonzero constant independent of n it follows that

$$\left| \frac{\Delta_{m,n+1}}{\Delta_{m-1,n}} \right|^{1/n} = \frac{\left| \dfrac{\Delta_{m,n+1}}{T^K_{m+1,n+1}} \right|^{1/n}}{\left| \dfrac{\Delta_{m-1,n}}{T^K_{m,n}} \right|^{1/n}} \left| \frac{T^K_{m+1,n+1}}{T^K_{m,n}} \right|^{1/n} \xrightarrow[n\to\infty]{} | a_{m+1} |^{-1}.$$

This proves (ii). From (9), (13) and (14) we conclude that

$$\left| Q_{m,n}(x) - \prod_{\nu=1}^{m} \left(1 - \frac{x}{a_\nu}\right)\right|$$

$$\leqslant 2 \sum_{k \neq K} | T^k_{m,n} |/| T^K_{m,n} | \, | T^K_{m,n}/\Delta_{m-1,n} | \prod_{\nu=1}^{m} \left(1 + \frac{R}{| a_\nu |}\right) \xrightarrow[n\to\infty]{} 0$$

for $| x | \leqslant R$, hence (iii) holds. Part (iv) follows by applying (ii) and (iii) to the sum (5) and using the argument at the end of the proof of Theorem 2.

Simple Sufficient Conditions

In this section we establish simple sufficient conditions such that the leading term of $\sum_k | T^k_{m,m} |$ is dominant; and hence that Theorem 2 holds.

Lemma 2. *Let $\{B_n\}$ be a sequence of positive real numbers. A necessary and sufficient condition that the leading term of the sum $\sum_{k \in S(m)} B_{k_1} B_{k_2} \cdots B_{k_m}$ be dominant as $m \to \infty$ is that $B_{m+1}/B_m \to 0$.*

Proof. Set

$$R_m = \sum_{i=1}^{\infty} \frac{B_{m+i}}{B_m} \qquad \text{and} \qquad T_m = R_m + R_m R_{m-1} + \cdots + R_m R_{m-1} \cdots R_1 .$$

Evidently,

$$0 < B_{m+1}/B_m < \sum_{k \neq K} B_{k_1} B_{k_2} \cdots B_{k_m} / B_1 B_2 \cdots B_m < T_m .$$

We need only show that $B_{m+1}/B_m \to 0$ implies that $T_m \to 0$. It is easy to see that our condition implies that $R_m \to 0$; hence choose $N \ni i \geqslant N$ implies $R_i < \frac{1}{2}$. Put $L = \max\{T_i : 1 \leqslant i \leqslant N\}$. Then for $m > N$,

$$T_m \leqslant \tfrac{1}{2} + (1/2^2) + \cdots + (1/2^{m-N})[R_N + R_N R_{N-1} + \cdots + R_N \cdots R_1]$$
$$< 1 + T_N \leqslant L + 1.$$

Therefore, $T_i \leqslant L + 1$ for all i and hence $0 < T_m = R_m(1 + T_{m-1}) \leqslant R_m(L + 2) \to 0$ as $m \to \infty$. This proves the lemma.

THEOREM 4. *Theorem* 2 *holds provided* $F(x)$ *satisfies*

(A) $|(A_{m+1}/A_m)(a_m/a_{m+1})^2| \to 0$ *as* $m \to \infty$,

and

(B) *There exists a constant* M *such that for all sufficiently large* m *and all* $k \in S(m)$ *we have* $M_m{}^k \leqslant M$, *where* $M_m{}^k$ *denotes the quantity*

$$\prod_{1 \leqslant i \leqslant j \leqslant m} |(a_{k_j}^{-1} - a_{k_i}^{-1})/(a_j^{-1} - a_i^{-1})|.$$

Proof. Let $B_n = |A_n/a_n{}^2|$. By the lemma,

$$\sum_{k \neq K} |T_{m,m}^k|/|T_{m,m}^K| \leqslant M^2 \sum_{k \neq K} |B_{k_1}B_{k_2} \cdots B_{k_m}/B_1B_2 \cdots B_m| \to 0 \text{ as } m \to \infty;$$

hence the result.

THEOREM 5. *Condition* B *of Theorem* 4 *holds, provided*

(a) $\limsup |a_n/a_{n+1}| = \gamma < 1$, *and*

(b) *for some* θ_0,

$$\limsup |\theta_n - \theta_0| < \tfrac{1}{2}\cos^{-1}(\gamma/2),$$

where $a_n = |a_n| \exp(i\theta_n)$ *and* $-\pi < \theta_n \leqslant \pi$.

Remark. Taking $a_n = n^2$ and $k = (1, 2,..., m-1, m+1)$ one finds that $M_m{}^k = 2(m+1)/(1+1/m)^{2m} \to \infty$ as $m \to \infty$ so the bound M does not exist in this case. Condition (b) enters in order to make use of the observation that if $|x| \leqslant \gamma < 1$ and $|\arg x| \leqslant \cos^{-1}(\gamma/2)$, then $|1 - x| \leqslant 1$.

Proof of the Theorem. We may choose γ_0 such that $\gamma < \gamma_0 < 1$ and $\limsup |\theta_n - \theta_0| < \frac{1}{2}\cos^{-1}(\gamma_0/2) < \frac{1}{2}\cos^{-1}(\gamma/2)$. Hence there exists N_0 such that $n \geqslant N_0$ implies $|a_n/a_{n+1}| \leqslant \gamma_0$ and $|\arg(a_i/a_j)| \leqslant \cos^{-1}(\gamma_0/2)$ for $j > i \geqslant N_0$. Since also $|a_i/a_j| \leqslant \gamma_0^{j-i} < \gamma_0$ it follows from the above remark that

$$1 - \gamma_0^{j-i} \leqslant |1 - (a_i/a_j)| \leqslant 1 \qquad \text{whenever} \quad j > i \geqslant N_0.$$

Choose $N_1 \geqslant N_0$ such that if $1 \leqslant i \leqslant N_0$ and $j \geqslant N_1$ then $\frac{1}{2} \leqslant |1-(a_i/a_j)| \leqslant 2$. Put $\mu = 4^{N_0}/\prod_{i=1}^{\infty}(1 - \gamma_0{}^i)$ and choose r such that $\gamma_0{}^r\mu \leqslant 1$. Then set $N = N_1 + r$ and $\lambda = \min_{1 \leqslant i < j \leqslant N} |a_j^{-1} - a_i^{-1}|$. We claim that a bound satisfying the requirements of the theorem is given by

$$M = \max\left\{\left(\frac{2}{\lambda |a_1|}\right)^{N(N-1)/2}, \mu^r\right\}.$$

Clearly M suffices when $m \leqslant N$. We use induction on m when $m \geqslant N$ and distinguish two cases.

Case 1. $k_{m-r+1} = m - r + 1$. Then $k_j = j$ for $1 \leqslant j \leqslant m - r + 1$. The factors in the numerator and denominator of the product M_{m+1}^k for which $1 \leqslant i < j \leqslant m - r + 1$ cancel, and upon writing out the remaining factors and using the above estimates for $|1 - (a_i/a_j)|$ one can show that

$$M_{m+1}^k \leqslant 2^{N_0 r} \Big/ \left(\frac{1}{2}\right)^{N_0 r} \left[\prod_{i=1}^{\infty} (1 - \gamma_0{}^i)\right]^r = \mu^r \leqslant M.$$

Hence M suffices in this case.

Case 2. $k_{m-r+1} \neq m - r + 1$. Then $k_j > j$ for $m - r + 1 \leqslant j \leqslant m + 1$. Using the induction hypothesis one can show that

$$M_{m+1}^k = M_m^{k'} \prod_{i=1}^{m} \left| \frac{a_{k_i}^{-1} - a_{k_{m+1}}^{-1}}{a_i^{-1} - a_{m+1}^{-1}} \right| \leqslant M \frac{2^{N_0} \left| \dfrac{a_{m-r+1}}{a_{k_{m-r+1}}} \right| \cdots \left| \dfrac{a_m}{a_{k_m}} \right|}{\left(\frac{1}{2}\right)^{N_0} \left| 1 - \dfrac{a_{N_0} + 1}{a_{m+1}} \right| \cdots \left| 1 - \dfrac{a_m}{a_{m+1}} \right|}$$

$$\leqslant M \frac{4^{N_0} \gamma_0^r}{\prod_{i=1}^{\infty} (1 - \gamma_0^i)} = M \gamma_0{}^r \mu \leqslant M.$$

This proves the theorem.

Summarizing, we have

THEOREM 6. *If* $F(x) = \sum_{i=1}^{\infty} (A_i/a_i - x)$ *satisfies the conditions*

(i) $|(A_{n+1}/A_n)(a_n{}^2/a_{n+1}^2)| \to 0$ *as* $n \to \infty$

(ii) $\limsup |a_n/a_{n+1}| = \gamma < 1$, *and*

(iii) *For some* θ_0, $\limsup |\theta_n - \theta_0| < \frac{1}{2}\cos^{-1}(\gamma/2)$, *then the sequence* $\{F_{m,m+p}(x)\}$, $p \geqslant 0$, *converge uniformly on any compact set bounded away from the poles of* $F(x)$. *Hence the P-fractions corresponding to the series* $x^{-k}F(x)$, $k \geqslant 0$, *converge.*

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